Tight Upper Bounds on the Redundancy of Optimal Binary AIFV codes

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Outline

1. Background

Binary Almost instantaneous FV (AIFV) codes

2. Main result:

Tight upper bounds on the redundancy of binary AIFV codes (worst-case redundancy)

3. Comparison with Huffman codes

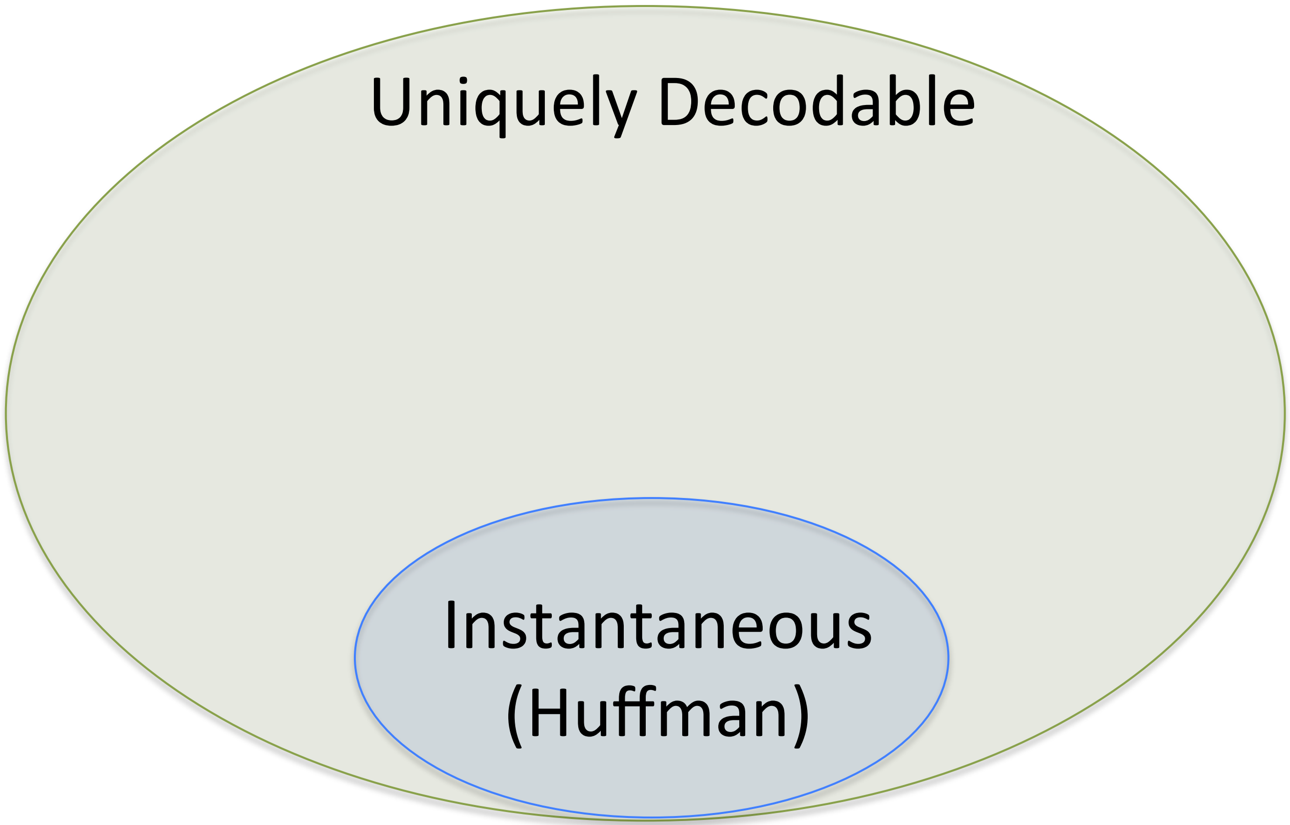
4. Idea & outline of proofs

5. Conclusion

Binary AIFV codes

Fixed-to-**Variable** length (FV) codes

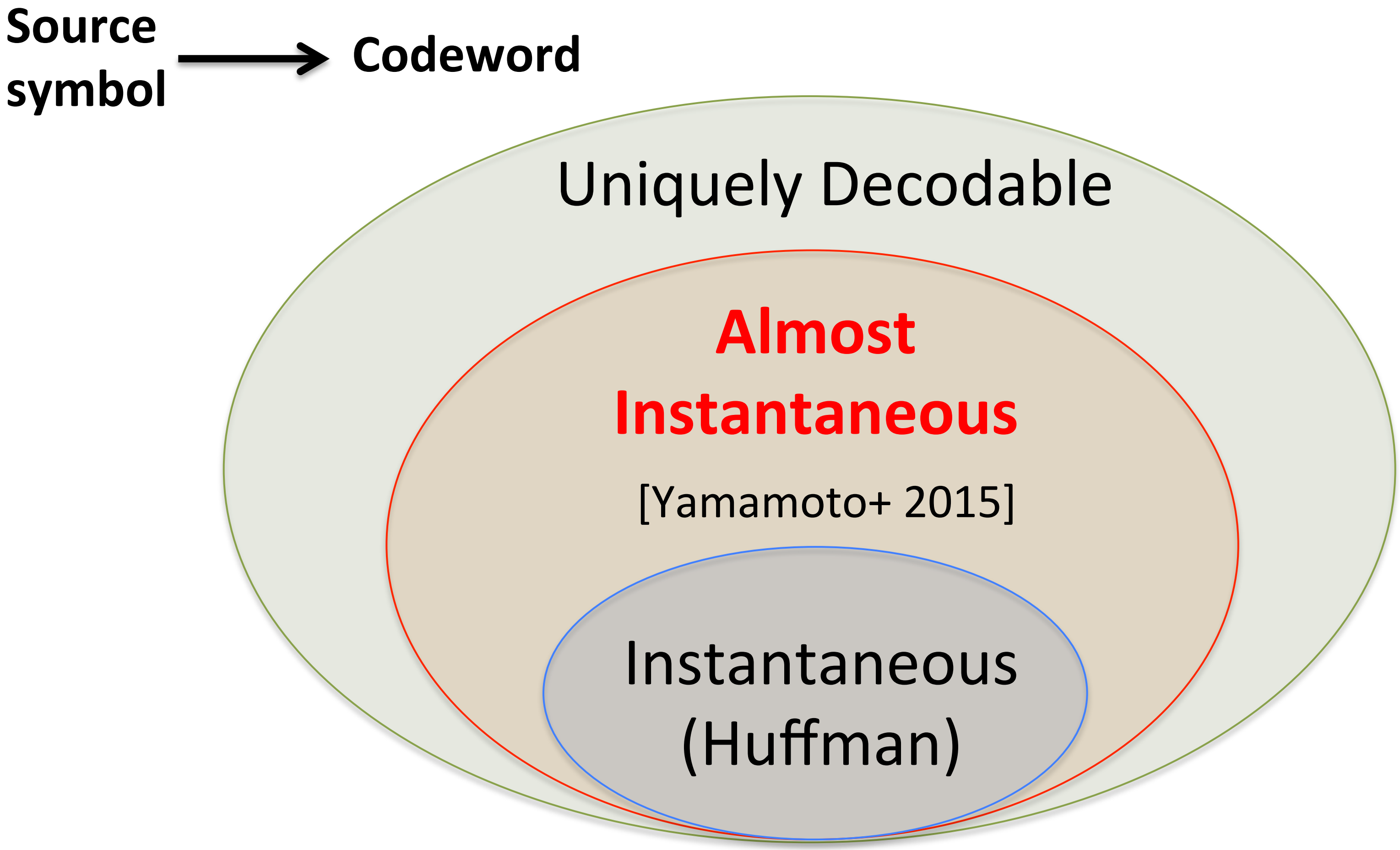
Source symbol → **Codeword**



Class of FV Codes

Binary AIFV codes

Fixed-to-**Variable** length (FV) codes



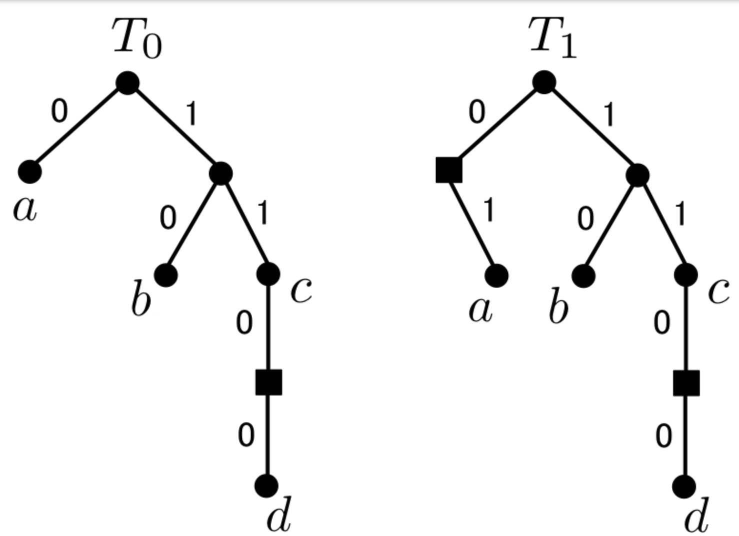
Class of FV Codes

Almost Instantaneous (AI) FV codes

Generalization of instantaneous binary FV codes

[Yamamoto, Tsuchihashi, Honda, 2015]

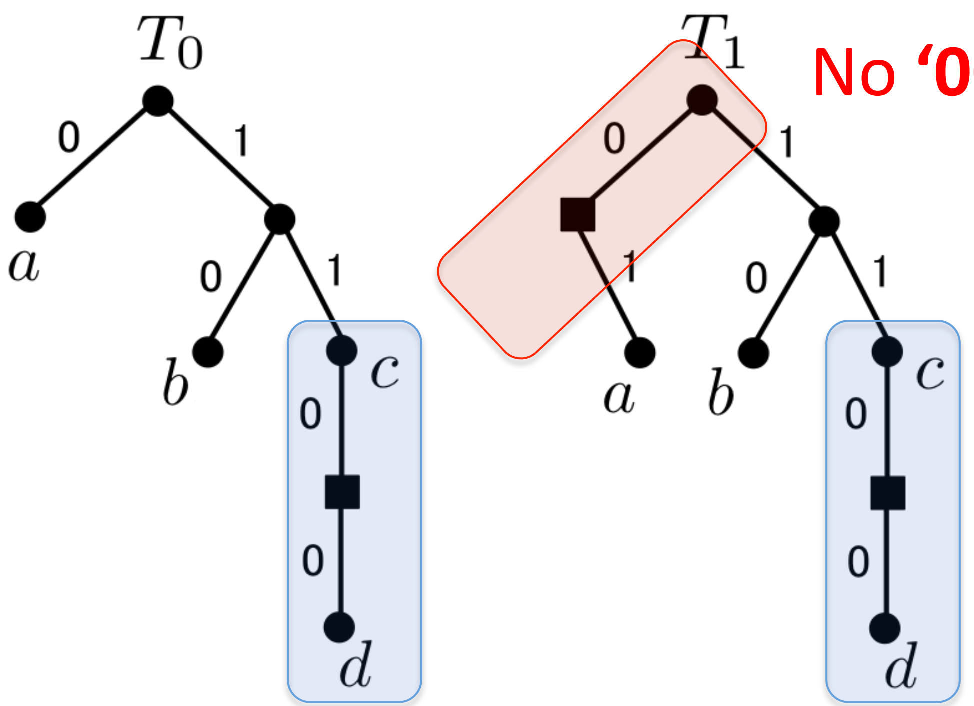
	Instantaneous	Almost Instantaneous
No.(Code Trees)	One	**Two**
Source Symbols	Leaves	**Leaves + incomplete node (master node)**
Decoding Delay	None	**At most 2 bits.**



Example of binary AIFV code trees.

Almost Instantaneous (AI) FV codes

Encoding (decoding) procedure use T_0 and T_1 iteratively.
After using a **master node**, use **T₁** for the next.



No **'00'** from the root.

Only **'00'** from the **master node.**

→The codes are **uniquely decodable.**

Worst-case Redundancy of AIFV codes

	Huffman code $\subset$	AIFV code
Redundancy	< 1	**< 1** [Yamamoto+ 2015]

AIFV codes have **good empirical performance**.
Even beat Huffman code for $\mathcal{X}^2$ for some sources.
[Yamamoto+ 2015]

Worst-case Redundancy of AIFV codes

	Huffman code $\subset$	AIFV code
Redundancy	< 1	**< 1** [Yamamoto+ 2015] **< 1/2 (Our result)**

$$p_{\max} \equiv \max_{x \in \mathcal{X}} p_X(x).$$

Worst-case redundancy in terms of $p_{\max}$ **(Our result)**

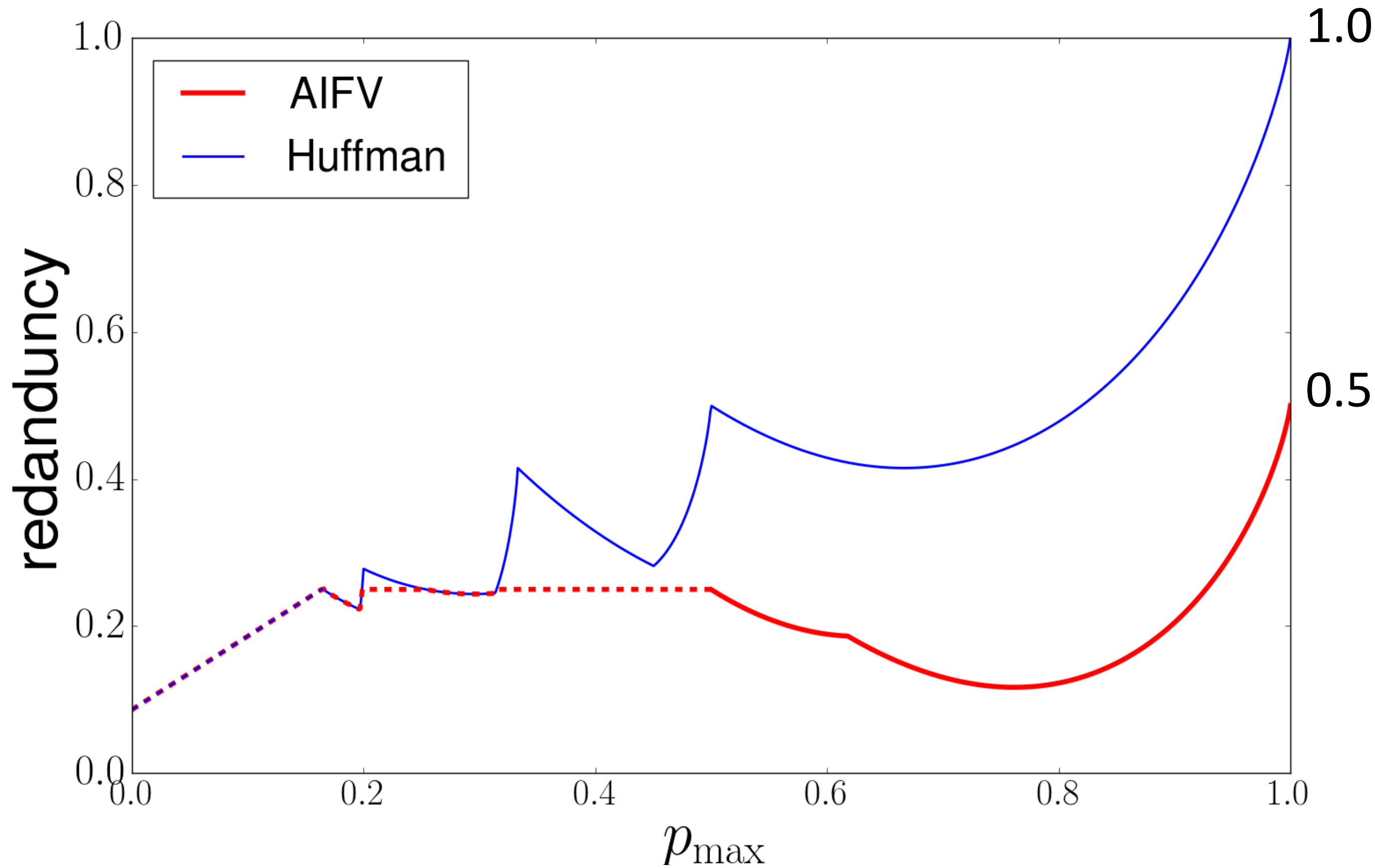
Worst-case Redundancy of AIFV codes

Theorem (Worst-case redundancy of AIFV codes)
For $p_{\max} = p \geq 1/2$, the worst-case redundancy of AIFV codes is

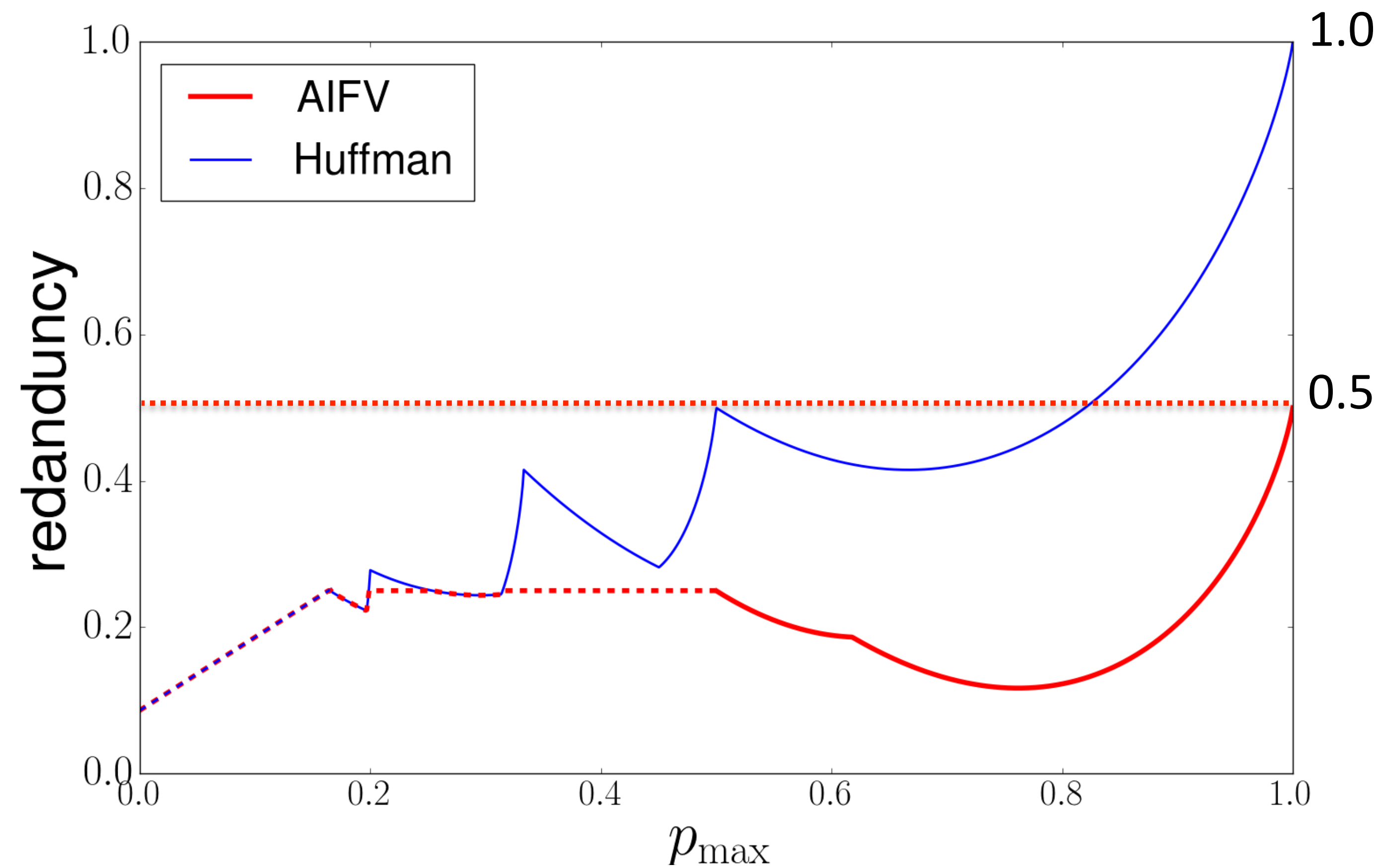
$$f(p) = \begin{cases} p^2 - 2p + 2 - h(p) & \text{if } \frac{1}{2} \leq p \leq \frac{-1+\sqrt{5}}{2}, \\ \frac{-2p^2+p+2}{1+p} - h(p) & \text{if } \frac{-1+\sqrt{5}}{2} \leq p < 1. \end{cases}$$

Theorem (Redundancy upper bound of AIFV codes)
For $p_{\max} < 1/2$, the worst-case redundancy is at most $\mathbf{1/4}$.

Comparison with Huffman codes



Comparison with Huffman codes



Corollary (Worst-case Redundancy)
Worst-case redundancy of binary AIFV codes is $\mathbf{1/2}$.

Comparison with Huffman codes

	Huffman	**AIFV**	Huffman for $\mathcal{X}^2$
Redundancy	< 1	**< 1/2**	< 1/2
Storage for Code trees	$\mathcal{O}(\|\mathcal{X}\|)$	$\mathcal{O}(\|\mathcal{X}\|)$	$\mathcal{O}(\|\mathcal{X}\|^2)$

$\mathcal{X}$: Source alphabet

More memory efficient than Huffman codes for $\mathcal{X}^2$.

Proof idea

Goal:

Prove bounds of optimal binary AIFV codes

Challenge:

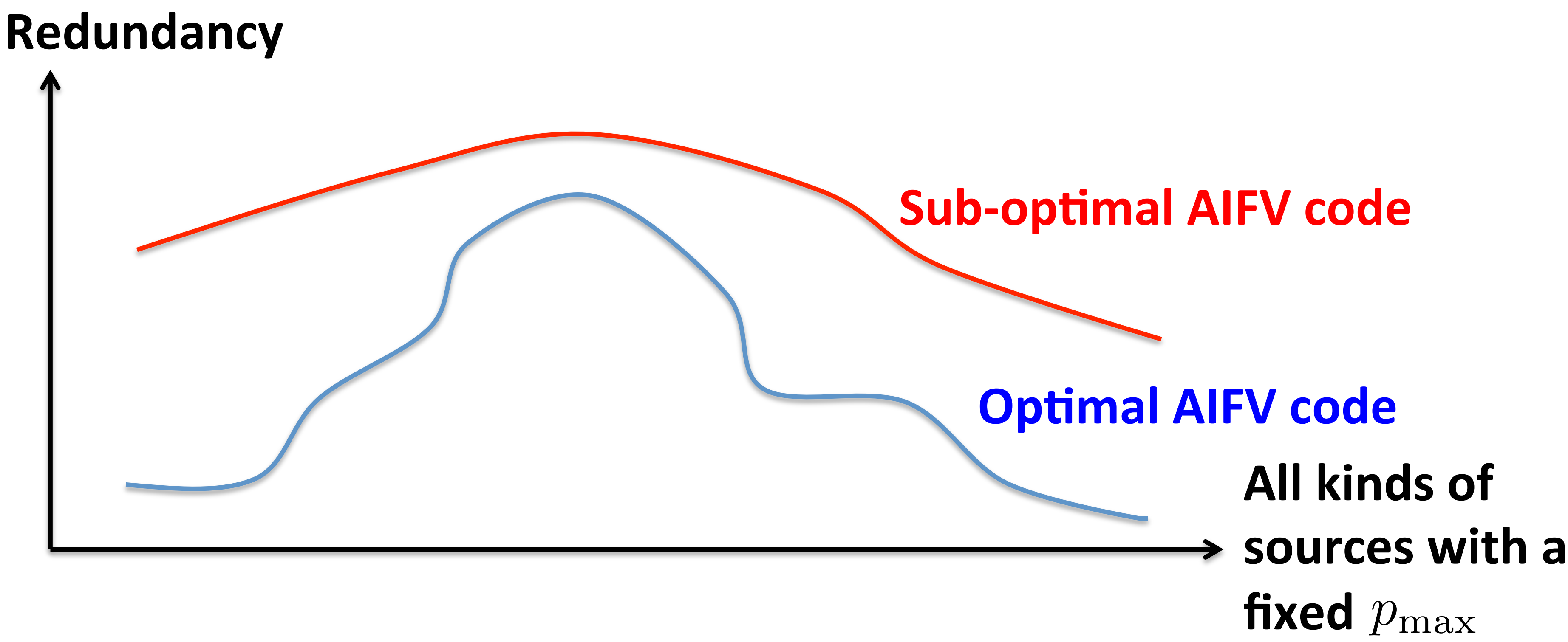
No simple algorithm known to construct the optimal AIFV code.

→ **Difficult to analyze** optimal code directly…

Proof idea

Our approach:

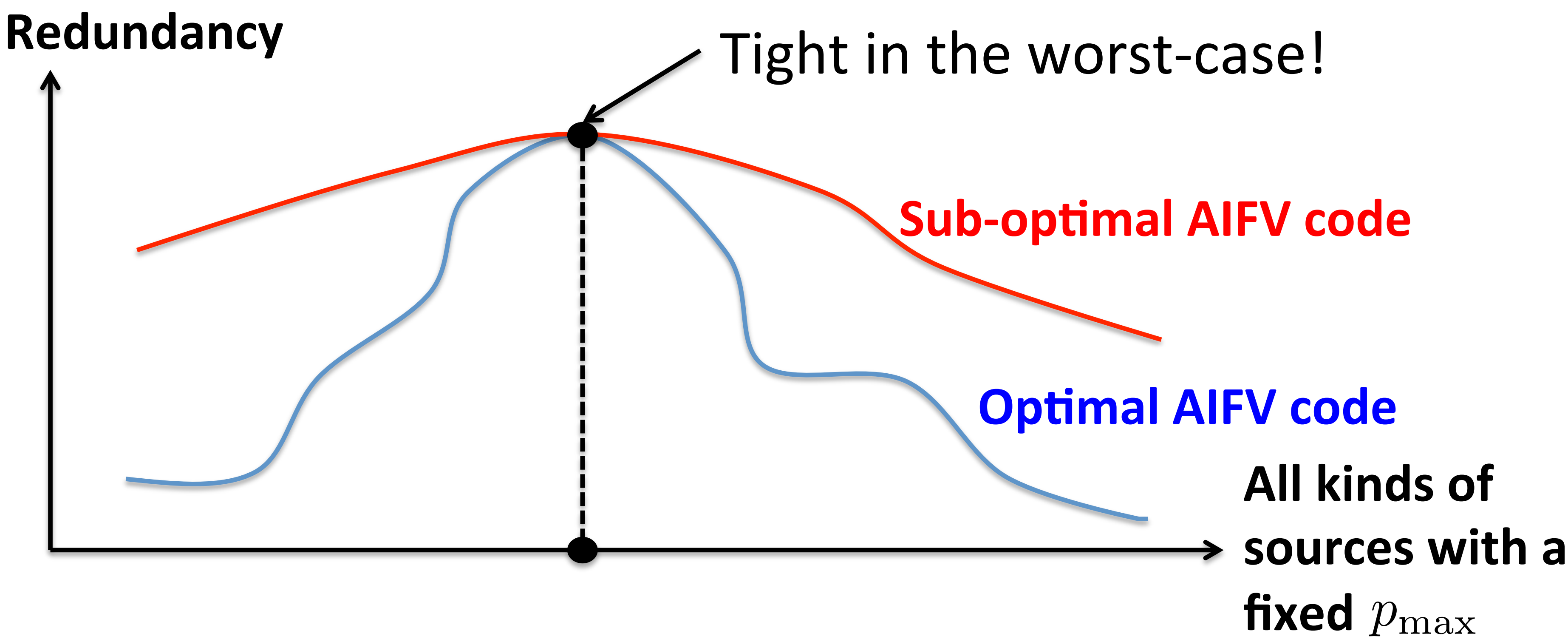
Simple Construction of **sub-optimal AIFV** codes from Huffman codes.



Proof idea

Our approach:

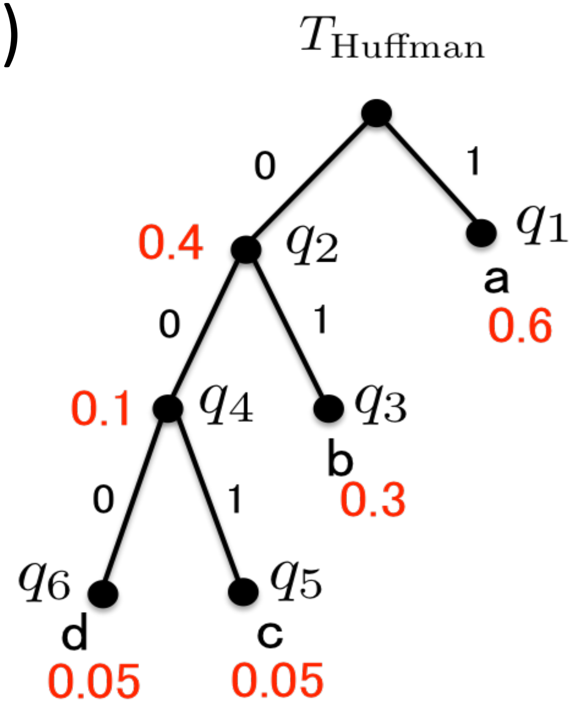
Simple Construction of **sub-optimal AIFV** codes from Huffman codes.



Proof outline (1/6)

- Simple **two-stage construction** of sub-optimal AIFV code trees from Huffman tree

Ex.)



Sibling property

[Gallager 1978]

$$q_1 \geq q_2, \cdots \geq q_{2K-2}$$

K : size of source alphabet

Sibling pair: $(q_{2k-1},\ q_{2k})$

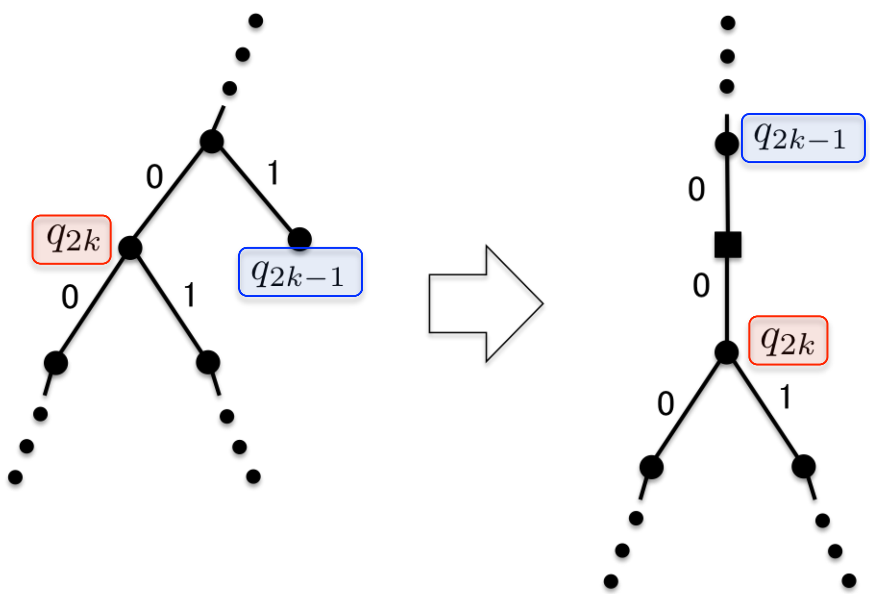
Proof outline (2/6)

- **Two-stage construction**

1. From T_{Huffman} to T_{base}

for sibling index $k = 2, \ldots, K-1$ **do**

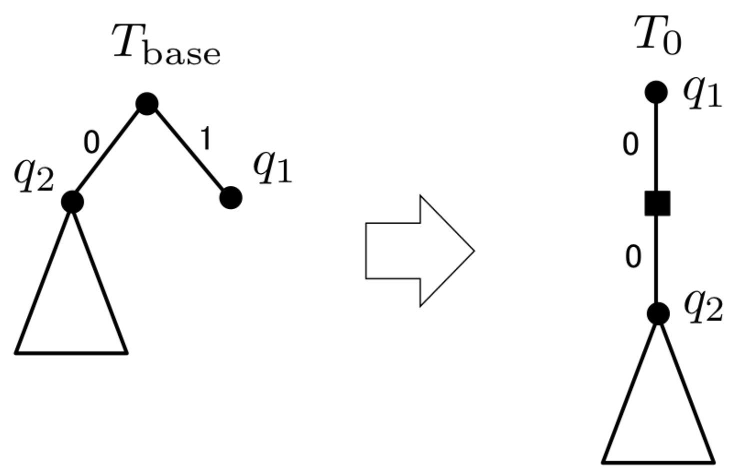
if $\boxed{q_{2k-1}}$ is a leaf and $\boxed{2q_{2k}} < \boxed{q_{2k-1}}$ **then**



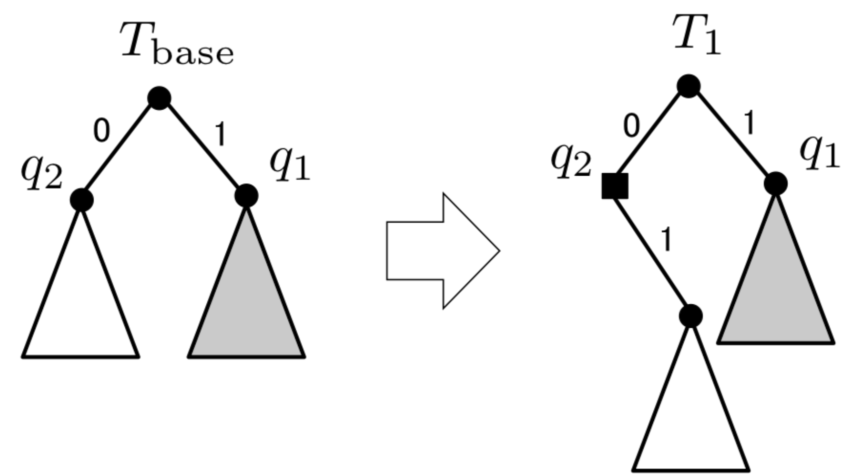
Proof outline (3/6)

- **Two-stage** construction

2. From T_{base} to T_0 and T_1



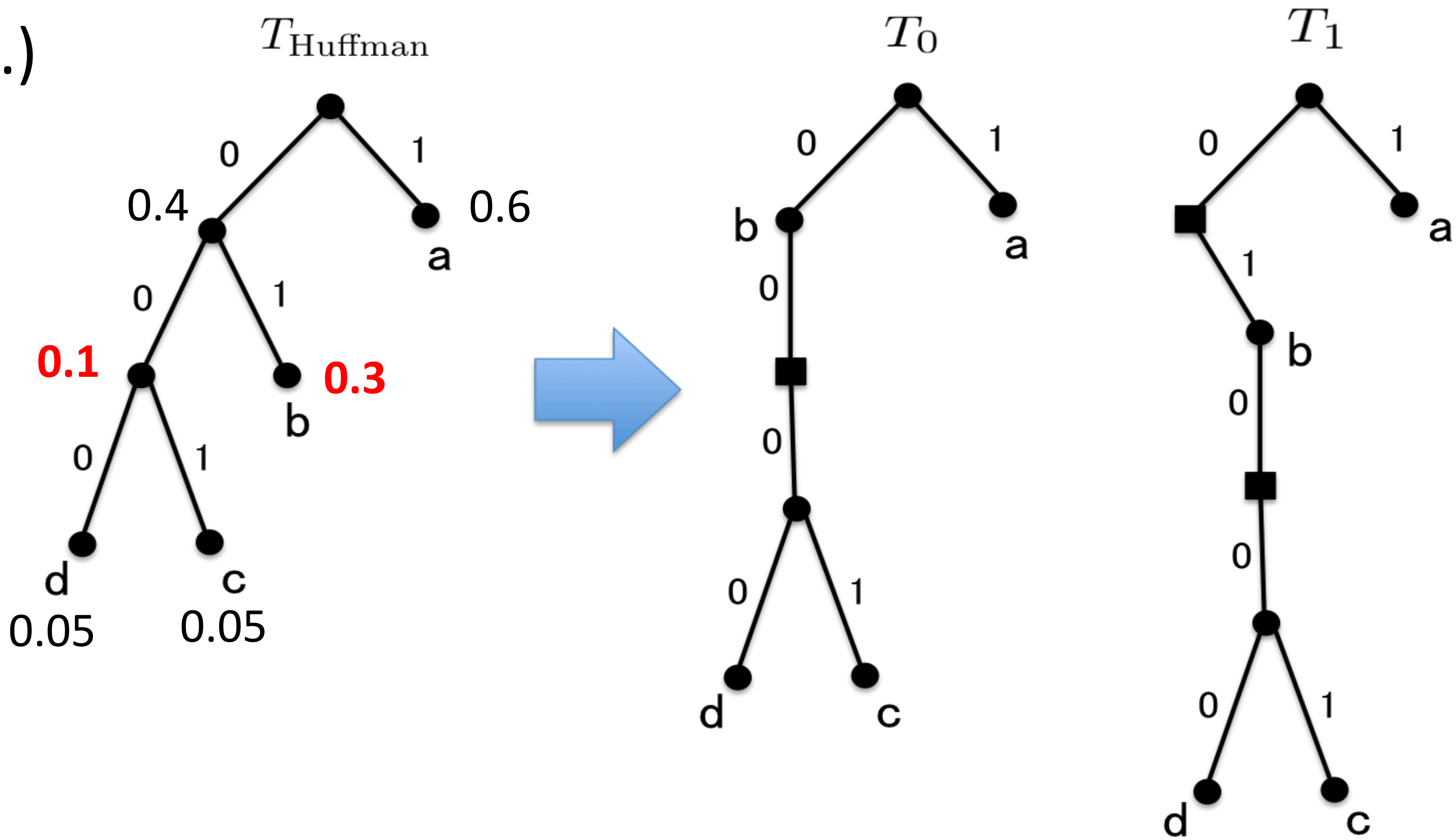
Only if $\frac{-1+\sqrt{5}}{2} \leq q_1$



Proof outline (4/6)

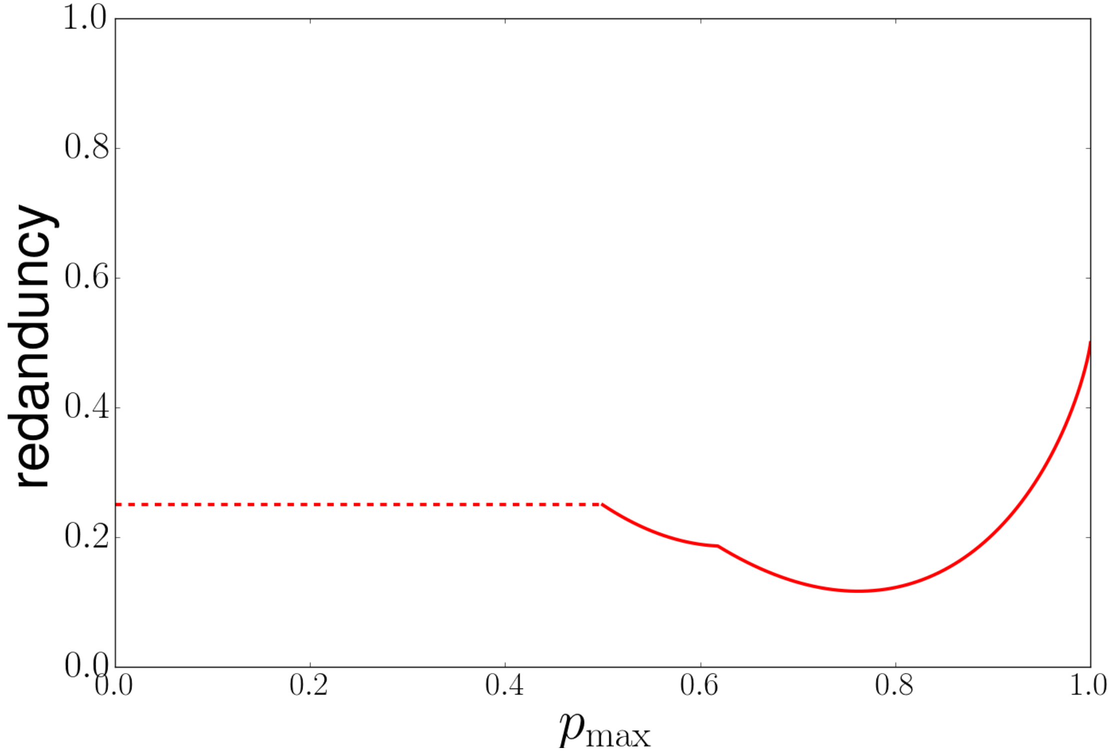
・Simple two-stage construction of sub-optimal AIFV code trees from Huffman tree

Ex.)



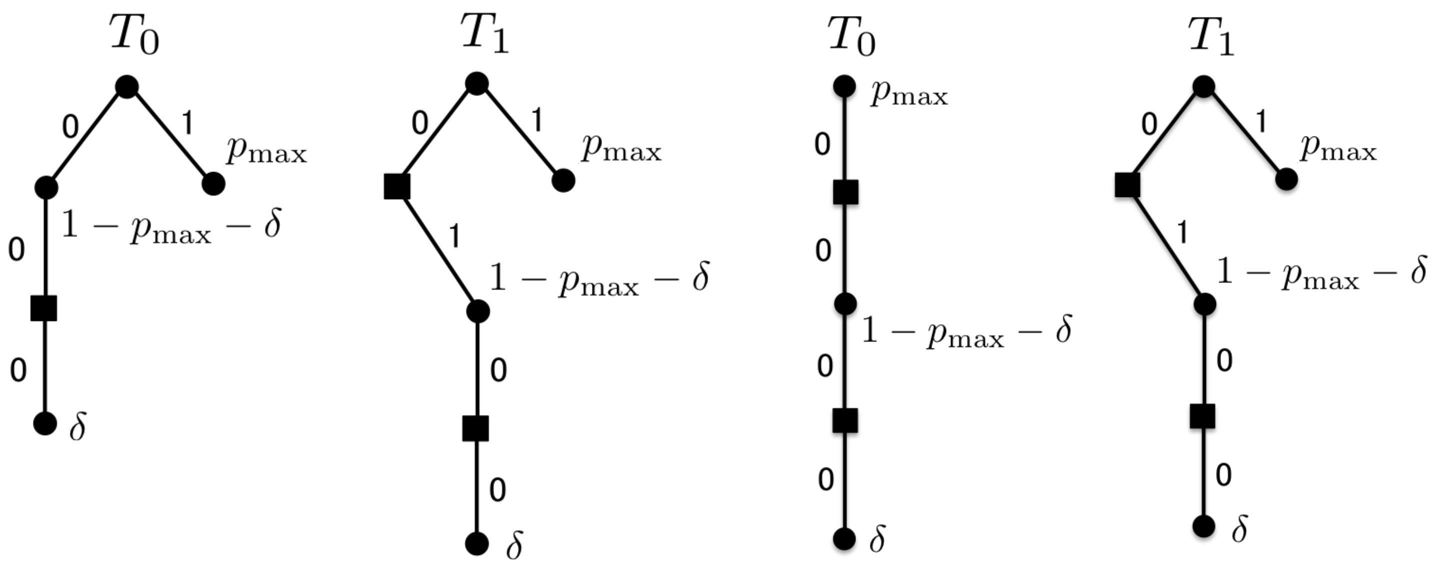
Proof outline (5/6)

- Upper bounds for sub-optimal AIFV code can be evaluated → tight for $p_{\max} \geq \frac{1}{2}$. **Why?**



Proof outline (6/6)

Optimal AIFV trees for $(p_{\max}, 1-p_{\max}-\delta, \delta)$ coincides with worst-case trees of sub-optimal AIFV codes.



(a) $\frac{1}{2} \leq p_{\max} \leq \frac{\sqrt{5}-1}{2}$. (b) $\frac{\sqrt{5}-1}{2} \leq p_{\max} \leq 1$.

→ The bound of sub-optimal trees is tight for $p_{\max} \geq \frac{1}{2}$.

Conclusion

1. Worst-case redundancy of binary AIFV codes is $\mathbf{1/2}$.
2. Worst-case redundancy in terms of $p_{\max} = p \geq 1/2$.

Theoretical justification for **superior performance of AIFV codes** over Huffman codes.

Further extension

If the codes are allowed to use **3 and 4 code trees**, worst-case redundancy is **1/3 and 1/4**, respectively.